Optimum Power-Limited Orbit Transfer in Strong Gravity Fields

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Analytic solutions are obtained for optimum transfer between arbitrary coplanar or coaxial ellipses in strong central gravitational fields. Fuel consumption is minimized for fixed transfer times with power-limited propulsion. The first-order secular solution obtained is valid only if many revolutions are made about the primary body during the transfer. The necessary conditions of Euler, Weierstrass, and Jacobi are satisfied so that the solutions actually furnish a local minimum. Multiple solutions are obtained for large changes in inclination or argument of perigee. The analysis shows which of these solutions furnishes an absolute minimum. It is found that all extremals violate the Jacobi condition and cease to be minimizing solutions if they are followed far enough.

Nomenclature

a = semimajor axis
e = eccentricity
K = gravitational constant
i = inclination of orbit plane
k_0, k_1, k_2 = constants of integration
s = dummy integration variable
U = variable defined by Eq. (32)
u = characteristic velocity
Θ = function defined by Eq. (24)
θ_1 = small rotation of major axis in orbit plane
θ_2 = small rotation of orbit plane around major axis
θ_3 = small rotation of orbit plane around latus rectum
$\theta_1(\tau), \theta_2(\tau)$ = solutions of Jacobi's differential equation
τ = variable defined by Eq. (21)
ϕ = arc-sine of the eccentricity
ψ = variable defined by Eq. (13)
Ω = longitude of the ascending node
ω = argument of perigee

Introduction

THE present paper is one of the series of papers[1–5] containing analytic solutions for optimum power-limited trajectories in inverse-square force fields. The previous papers in this series have utilized two different approximations. The first approximation, which is used in this paper, is that the transfer time is long compared to the period of an elliptic orbit. The papers with this assumption have considered transfer between inclined circular orbits,[1] escape from elliptic orbits,[2] and transfer between coplanar circular and elliptic orbits.[3] The second approximation has been that changes in all the elements are small so that transfer is made to a neighboring orbit. The papers utilizing this approximation have considered transfer between neighboring circular orbits[4] and transfer and rendezvous between neighboring elliptic orbits.[5]

The propulsion system that effects the changes of the elliptic orbit is assumed to be power limited, i.e., it operates at constant exhaust power with a thrust magnitude inversely proportional to the exhaust velocity.[6, 7] The direction and magnitude of the thrust, both of which are assumed to be completely variable, is to be determined as a function of time so as to minimize the fuel consumption for a fixed total transfer time.

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There have been a number of previous studies of trajectory optimization for power-limited propulsion systems. Some of the more important studies are contained in Refs. 6–14. Most of these studies are concerned with numerical results for inverse-square force fields.[6–12] Reference 13 reports analytic solutions for optimum trajectories in field-free space, whereas Ref. 14 contains analytic solutions for the optimum thrust programs for small changes of five elements of an elliptic orbit in an inverse-square field. Reference 5 determines the effect of these optimum thrust programs upon the elements of the orbit and determines the minimum fuel consumption for small changes in all the elements of elliptic orbits. The present paper extends the analysis of Ref. 5 to determine fuel requirements for large changes in several of the elements of elliptic orbits with the approximation that these changes are made slowly. In particular, the general coplanar and coaxial transfers between elliptic orbits is determined in the first approximation.

The method of averaging used in the present paper is also utilized in Refs. 1–3. It falls into the category that Bogoliubov and Mitropolsky[15] call the method of rapidly-rotating phase. The evaluation is carried out only as far as the first-order secular terms, although these terms can be combined with the first-order periodic terms derived in Ref. 5 to produce an improved first approximation. In Ref. 16, Iszak has recently pointed out that the method of rapidly-rotating phase is equivalent to the Von Zeipel method used in celestial mechanics.[17]

Analysis

Reference 5 contains a derivation of both the secular and periodic terms for small changes in elliptic orbits and an explicit expression for the minimum fuel consumption considering only the secular terms. This expression, given in Eq. (1), is the starting point of the present paper:

$$\Delta u = \left(\frac{K}{a}\right)^{1/2} \left(\frac{\Delta a^2}{4a^2} + \frac{2}{5}\,\Delta\phi^2 + \frac{2}{1 + 5\cot^2\phi}\,\Delta\theta_1^2 + \frac{2}{1 + 5\tan^2\phi}\,\Delta\theta_2^2 + 2\Delta\theta_3^2\right)^{1/2} \quad (1)$$

where

$$\Delta\theta_1 \equiv \Delta\omega + \Delta\Omega \cos i$$
$$\Delta\theta_2 \equiv \Delta i \cos\omega + \Delta\Omega \sin i \sin\omega$$
$$\Delta\theta_3 \equiv -\Delta i \sin\omega + \Delta\Omega \sin i \cos\omega$$
$$\phi \equiv \sin^{-1} e$$

The variable u is essentially a characteristic velocity. It represents the maximum velocity that could be obtained in field-free space in the same time with the same propulsion system and fuel consumption. In Ref. 3 it was shown that, for long-time transfers, minimizing u is equivalent to minimizing the fuel consumption and that during such transfers the average value of the square of the optimum thrust acceleration remains constant. The three variables θ_1, θ_2, and θ_3 represent small orthogonal rotations around the polar axis of the orbit, around the major axis of the orbit, and around the latus rectum of the orbit. Their equivalent expressions in terms of the conventional Euler angles are shown following Eq. (1). The variable ϕ, a variable used by Plummer,[18] is the arc sine of the eccentricity.

The variational problem is formulated in Eq. (2) as a problem of simplest type in parametric form. The independent variable s is merely a dummy variable of integration. The dots indicate differentiation with respect to s:

$$u = \int \left(\frac{K}{a}\right)^{1/2} \left(\frac{\dot{a}^2}{4a^2} + \frac{2}{5}\dot{\phi}^2 + \frac{2\dot{\theta}_1^2}{1 + 5\cot^2\phi} + \frac{2\dot{\theta}_2^2}{1 + 5\tan^2\phi} + 2\dot{\theta}_3^2\right)^{1/2} ds = \int f(y, \dot{y})ds \quad (2)$$

This problem could be interpreted geometrically as a problem of minimum arc length in a five-dimensional phase space. Because the optimum thrust programs for changing each of the five elements of the orbit produce no secular changes in the other four elements, there are no cross-product terms in Eq. (2), and the phase space is an orthogonal phase space. The first necessary condition for a minimum in this problem is the Euler-Lagrange equation:

$$(d/ds)f_{\dot{y}_i} - f_{y_i} = 0 \qquad i = 1, \ldots, 5 \quad (3)$$

A first integral of Eq. (3) may be found by noting that the same power of the semimajor axis appears in all but the first term under the second square root sign in Eq. (2). This feature allows the definition of a new payoff variable with Eq. (4):

$$\dot{\psi} \equiv \left(\frac{\dot{\phi}^2}{5} + \frac{\dot{\theta}_1^2}{1 + 5\cot^2\phi} + \frac{\dot{\theta}_2^2}{1 + 5\tan^2\phi} + \dot{\theta}_3^2\right)^{1/2} \quad (4)$$

In terms of this variable, Eq. (2) may be rewritten as follows:

$$u = \int \left(\frac{K}{4}\frac{\dot{a}^2}{a^3} + 2\frac{K}{a}\dot{\psi}^2\right)^{1/2} ds \quad (5)$$

The Euler-Lagrange equation for Eq. (5) is given by,

$$\frac{d}{ds}\left(\frac{2(K/a)\dot{\psi}}{[K(\dot{a}^2/4a^3) + 2(K/a)\dot{\psi}^2]^{1/2}}\right) = 0 \quad (6)$$

Equation (6) may be integrated once to yield Eq. (7), where k_0 is a constant of integration, and a_0 is the initial value of the semimajor axis:

$$\dot{\psi} = \dot{a}/[(8)^{1/2}(aa_0 \csc^2 k_0 - a^2)^{1/2}] \quad (7)$$

Equation (7) may then be integrated to yield,

$$a/a_0 = \sin^2[(2)^{1/2}\psi + k_0]/\sin^2 k_0 \quad (8)$$

the equation of the extremals, and Eq. (8) may be substituted back into Eq. (5) and integrated to yield the value of the payoff as follows:

$$u = (K/a_0)^{1/2}[\sin(2)^{1/2}\psi/\sin(2)^{1/2}\psi + k_0)] \quad (9)$$

It is possible to eliminate the value of k_0 between Eqs. (8) and (9) and to solve directly for the fuel consumption required for any transfer:

$$u = \left(\frac{K}{a_0}\right)^{1/2}\left[1 - 2\left(\frac{a_0}{a_1}\right)^{1/2}\cos(2)^{1/2}\psi + \frac{a_0}{a_1}\right]^{1/2} \quad (10)$$

Equations (8) and (9) are plotted in Fig. 1, which shows the extremals and the transversals (the constant payoff lines) for this problem. Each extremal is characterized by its value of the constant k_0. If only ψ is to be changed, the trajectory becomes first larger and then smaller because it is easier to change ψ when the orbit is larger.

Equations (8–10) represent a complete solution for the problems considered in Refs. 1 and 3. The problem of Ref. 1 corresponds to the definitions given in Eq. (11), and the problem of Ref. 3 corresponds to the definitions given in Eq. (12):

$$\phi \equiv 0 \qquad \omega = 0 \qquad \Omega = 0 \qquad \psi = i \quad (11)$$

$$\omega \equiv 0 \qquad \Omega \equiv 0 \qquad i \equiv 0 \qquad \psi = \phi/(5)^{1/2} \quad (12)$$

The first integral of the problem given by Eq. (10) reduces the general five-dimensional problem to the four-dimensional problem given by Eq. (13) with a new payoff variable ψ:

$$\psi = \int \left(\frac{\dot{\phi}^2}{5} + \frac{\dot{\theta}_1^2}{1 + 5\cot^2\phi} + \frac{\dot{\theta}_2^2}{1 + 5\tan^2\phi} + \dot{\theta}_3^2\right)^{1/2} ds \quad (13)$$

The solution of this general four-dimensional problem is quite difficult and is not attempted here. Instead, two two-dimensional problems are solved: the coplanar problem given by Eq. (14) and the coaxial problem given by Eq. (15):

$$\psi = \int \left(\frac{\dot{\phi}^2}{5} + \frac{\dot{\omega}^2}{1 + 5\cot^2\phi}\right)^{1/2} ds \qquad i = \Omega \equiv 0 \quad (14)$$

$$\psi = \int \left(\frac{\dot{\phi}^2}{5} + \frac{\dot{i}^2}{1 + 5\tan^2\phi}\right)^{1/2} ds \qquad \omega = \Omega \equiv 0 \quad (15)$$

These two problems can be transformed into each other by means of the following transformation:

Coplanar		Coaxial
a	$\longleftrightarrow$	a
ϕ	$\longleftrightarrow$	$\phi - \tau/2$
ω	$\longleftrightarrow$	i
$i = 0$	$\longleftrightarrow$	$\omega = 0$
$\Omega = 0$	$\longleftrightarrow$	$\Omega = 0$

(16)

These two problems are completely equivalent, and it is only necessary to solve one of them to obtain the solution of both. Since some choice must be made, the coplanar problem will be the one solved in detail. The Euler-Lagrange equation of this problem is given by Eq. (17):

$$\frac{d}{ds}\left\{\frac{\dot{\omega}/(1 + 5\cot^2\phi)}{[(\dot{\phi}^2/5) + \dot{\omega}^2/(1 + 5\cot^2\phi)]^{1/2}}\right\} = 0 \quad (17)$$

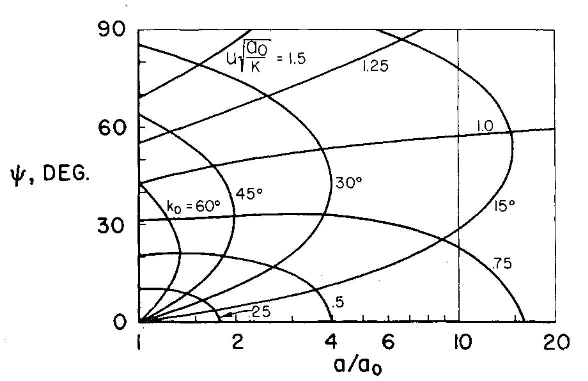

Fig. 1 Extremals and payoff curves for changes in the major axis.

Its first integral is given in Eq. (18) with k_1 being a constant of integration:

$$\dot{\omega} = \frac{1 + 5\cot^2\phi}{5(\cot^2 k_1 - \cot^2\phi)^{1/2}}\dot{\phi} \tag{18}$$

Equation (18) may then be integrated to yield Eq. (19), where k_2 is a second constant of integration:

$$\omega = k_2 + \cos^{-1}\left(\frac{\cot\phi}{\cot k_1}\right) - \frac{4}{5}\sin k_1 \cos^{-1}\frac{\cos\phi}{\cos k_1} \tag{19}$$

Substituting Eq. (19) back into Eq. (14) and integrating, Eq. (20) may be obtained:

$$\psi = k_3 + \frac{(1 + 4\cos^2 k_1)^{1/2}}{5}\cos^{-1}\left(\frac{\cos\phi}{\cos k_1}\right) \tag{20}$$

The Euler-Lagrange equations are only one of several necessary conditions for a minimization of the integral of Eq. (14). A second necessary condition, that of Jacobi, is now investigated. For this investigation, it is convenient to define a new variable τ by Eq. (21):

$$\tau \equiv (\psi - k_3)[5/(1 + 4\cos^2 k_1)^{1/2}] \tag{21}$$

In terms of this variable Eqs. (19) and (20) become a parametric representation of the solution given by Eqs. (22) and (23):

$$\omega = f(\tau, k_1, k_2) = k_2 + \tan^{-1}(\tan\tau \csc k_1) - \tfrac{4}{5}\tau \sin k_1 \tag{22}$$

$$\cos\phi = g(\tau, k_1, k_2) = \cos k_1 \cos\tau \tag{23}$$

Jacobi's condition is stated in terms of the solution of a certain differential equation, Jacobi's equation.[19] In Ref. 19 it is shown that solutions of this equation $[\theta_1(\tau), \theta_2(\tau)]$ may be found by simply differentiating the solutions of the Euler-Lagrange equation [Eq. (24)]. The Jacobi condition may then be stated; the Θ function of Eq. (25) does not vanish along the trajectory of interest (Eq. 26):

$$\theta_1(\tau) = g_\tau(\tau)f_{k_1}(\tau) - f_\tau(\tau)g_{k_1}(\tau)$$

$$\theta_2(\tau) = g_\tau(\tau)f_{k_2}(\tau) - f_\tau(\tau)g_{k_2}(\tau) \tag{24}$$

$$\Theta(\tau, \tau_0) = \theta_1(\tau)\theta_2(\tau_0) - \theta_2(\tau)\theta_1(\tau_0) \tag{25}$$

$$\Theta(\tau, \tau_0) \neq 0 \qquad \tau_0 < \tau < \tau_1 \tag{26}$$

For the present problem Eq. (24) may be written explicitly as Eq. (27) and the Θ function becomes Eq. (28):

$$\theta_1(\tau) = \cos\tau(1 - \tfrac{4}{5}\sin^2 k_1) + \tfrac{4}{5}\tau\sin\tau\cos^2 k_1$$

$$\theta_2(\tau) = -\cos k_1 \sin\tau \tag{27}$$

$$\Theta = (1 - \tfrac{4}{5}\sin^2 k_1)\sin(\tau - \tau_0) - \tfrac{4}{5}\cos^2 k_1(\tau - \tau_0)\sin\tau\sin\tau_0 \tag{28}$$

The first point at which this equation vanishes after the initial point of an extremal is called a conjugate point. A conjugate point may be interpreted geometrically as the first point of tangency of an extremal with an envelope of extremals. It is shown in Ref. 19 that an extremal ceases to be a minimizing solution at a conjugate point.

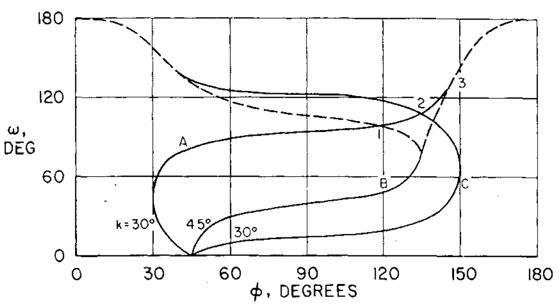

Fig. 2 Structure of the extremals for $\phi_0 = 45°$.

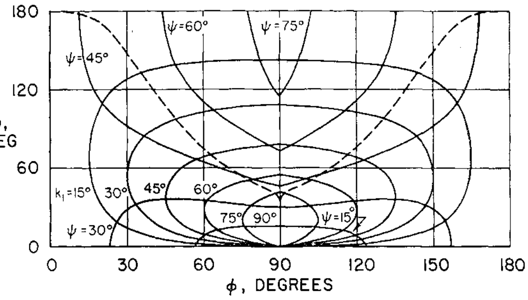

Fig. 3 Extremals and payoff curves for coplanar transfer with $\phi_0 = 90°$.

Every extremal of the present problem has a conjugate point corresponding to a root of Eq. (28). The character of the extremals and the conjugate points may be seen from Fig. 2. This figure is a plot of the argument of perigee vs the arc sine of the eccentricity and shows three extremals A, B, and C. The dotted lines represent the envelope of the extremals and correspond to the boundary between the region where there is only one extremal through a given point from the initial point and the region where there is more than one such extremal. If extremal A is followed from its initial point, zero, it first enters the region of multiple solutions when it crosses the envelope at point 1. At point 2, extremal A has the same payoff ψ as does extremal C. Up to point 2, extremal A is not only a local minimum but is also the absolute minimum solution to any point along its path. Between points 2 and 3, extremal A is still a local minimum but there are other extremals that provide lower values of ψ. At point 3, extremal A first becomes tangent to the envelope of the extremals and has a conjugate point. Beyond this point, extremal A ceases to be a minimizing solution, and better trajectories may be found in the neighborhood of A.

Extremal B is a critical extremal that divides the phase space into two sets of extremals. For extremal B, points 1, 2, and 3 coincide, and the conjugate point occurs where extremal B ceases to become both an absolute minimum and a local minimum. The root of Eq. (28) for this critical extremal is given by the expressions of Eq. (29).

$$k_1 = \phi_0 \qquad \tau_0 = 0° \qquad \tau = 180°$$
$$\phi_1 = \pi - \phi_0 \qquad \omega_1 = \pi(1 - \tfrac{4}{5}\sin\phi_0) \tag{29}$$

The complete family of extremals and transversals given by Eqs. (19) and (20) is plotted in Figs. 3–5 for values of ϕ_0 of 90°, 45°, and 0°. For the 90° case shown in Fig. 3, the initial orbit is a unit-eccentricity ellipse, a straight line of finite length. The values of ϕ less than 90° correspond to elliptic orbits having one sense of rotation, while the values of ϕ greater than 90° correspond to elliptic orbits

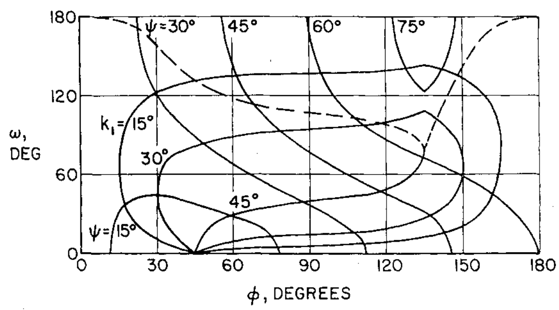

Fig. 4 Extremals and payoff curves for coplanar transfer with $\phi_0 = 45°$.

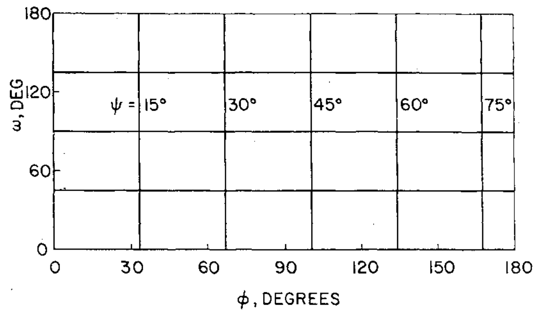

Fig. 5 Extremals and payoff curves for coplanar transfer in the degenerate case $\phi_0 = 0°$.

having the other sense of rotation. For this case, the problem and consequently Fig. 3 are completely symmetrical. Once again the first envelope of the extremals is shown by the two dotted lines. The particular case in which the final orbit is also of unit eccentricity provides a good illustration of the significance of the conjugate point. For changes in the argument of perigee of less than 36°, the optimum transfer orbits remain of unit eccentricity, and the optimum trajectory in the ω, ϕ phase plane is a vertical line. Above 36° this critical extremal has a conjugate point, and the optimum transfer involves orbits that first decrease their eccentricity and then increase it. The maximum decrease in eccentricity is a function of the total rotation of the major axis. For the particular case where the rotation angle is 180° so that the orbit faces in the other direction, the eccentricity of the transfer orbit decreases to zero.

The coaxial problem corresponding to this coplanar problem is the more practical problem of transfer between inclined circular orbits. When the angle between the initial and final orbits is less than 36°, the intermediate orbits remain circular and the solution given in Ref. 1 may be used. When this angle is greater than 36°, the optimum intermediate orbits become elliptical and the solutions of the present paper should be used.

Figure 4 shows similar results for an initial value of ϕ of 45°. As can be seen, these trajectories are merely a distortion of the trajectories shown in Fig. 3, and their general properties remain the same. Finally, Fig. 5 shows the case for ϕ_0 equal to 0° where the initial orbit is circular. In this case the optimum trajectory always corresponds to rotating the major axis through the appropriate angle first and then increasing the eccentricity to the desired value. Here, both the extremals and the transversals are straight lines.

In order to show that the solutions obtained actually do minimize the payoff, it is still necessary to consider the Weierstrass and Legendre conditions. A particularly simple form of the Weierstrass condition given in Ref. 19 is as follows:

$$F_1 = \frac{1}{5(1 + 5\cot^2\phi)\left(\dfrac{\dot\phi^2}{5} + \dfrac{\dot\omega^2}{1 + 5\cot^2\phi}\right)^{1/2}} \geq 0 \tag{30}$$

If the F_1 function is always greater than zero for bounded values of $\dot\phi$ and $\dot\omega$, the Weierstrass condition is satisfied in the strong form. It can be seen that this condition will always be satisfied in the strong form as long as ϕ is not zero. When ϕ is equal to zero, F_1 is equal to zero, and the Weierstrass condition is satisfied only in its weak form. The satisfaction of the Weierstrass condition automatically satisfies the weaker condition of Legendre.

The satisfaction of the four necessary conditions including the Weierstrass condition in its strong form is sufficient[19] for a local minimum of the solutions obtained to the coplanar problem given by Eq. (14). It still remains to prove sufficiency for the problem of Eq. (5). The fact that this particular three-dimensional problem can be broken down into two two-dimensional problems is an appreciable simplification due to the cylindrical nature of the phase space. The Weierstrass condition for the variational problem of Eq. (5) is given by Eq. (31) and is seen to be satisfied in the strong form as long as the major axis does not become infinite:

$$F_1 = \frac{K^2/2a^4}{[K(\dot a^2/4a^3) + 2(K/a)\dot\psi^2]^{1/2}} \geq 0 \tag{31}$$

Finally, to prove sufficiency for the extremals that solve Eq. (4), it is still necessary to check the Jacobi condition. This is done by defining a new independent variable with Eq. (32) and rewriting Eqs. (8) and (9) as Eq. (33):

$$U \equiv (a_0/K)^{1/2}u \tag{32}$$

$$a = f(U, a_0, k_0) = \frac{a_0}{1 - 2U\cos k_0 + U^2}$$

$$\tan(2)^{1/2}\psi = g(U, a_0, k_0) = \frac{U \sin k_0}{1 - U\cos k_0} \tag{33}$$

The two solutions of Jacobi's equation derived from Eqs. (33) are given by Eqs. (34) and finally the Θ function is given by Eq. (35):

$$\theta_1(U) = \frac{\sin k_0}{(1 - U\cos k_0)^2(1 - 2U\cos k_0 + U^2)}$$

$$\theta_2(U) = \frac{-2a_0U}{(1 - U\cos k_0)^2(1 - 2U\cos k_0 + U^2)} \tag{34}$$

$$\Theta = \frac{2a_0 \sin k_0(U - U_0)}{(1 - U\cos k_0)^2(1 - 2U\cos k_0 + U^2)} \tag{35}$$

It is readily seen that Eq. (35) can have no zeros for any value of U greater than U_0 so that the extremals of this problem can never have a conjugate point. The extremals given by Eqs. (8) and (9) actually minimize Eq. (5).

In summary, this paper has considered the first-order secular solution for the optimum power-limited transfer between elliptic orbits. A new first integral of the general five-dimensional problem has been found which reduces it to a four-dimensional problem. The particular cases of coplanar and coaxial elliptic orbits have been solved in detail. The minimizing extremal to every point in the phase plane has been identified and the sufficiency of these solutions has been established. It has also been shown that every extremal has a conjugate point if followed far enough.

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Quadratic and Higher-Order Feedback Gains for Control of Nonlinear Systems

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The problem of controlling a nonlinear system optimally in the presence of deterministic disturbances is treated. In particular, optimal feedback control in the vicinity of an optimal nominal trajectory is sought. The control law is to preserve the optimality of the nominal path and provide terminal control to the nominal end point defined by hard constraints on functions of the terminal state. A method of obtaining approximations of any order to such a nonlinear optimal feedback control law is presented. Linear differential equations for the linear and quadratic feedback gains are explicitly given. The equations for the linear gains are homogeneous, whereas those for higher-order gains are nonhomogeneous. The forcing terms are functions of the gains of the next lower order. Closed-form expressions for the linear and quadratic gains for a simple intercept problem and their simulations are presented. The addition of quadratic gains to the linear control approximation generally results in improved control.

I. Introduction and Summary

MOST methods developed to date for the solution of nonlinear optimal feedback control problems are based on linearization about some reference trajectory of the basic nonlinear system. As such, they provide a linear feedback control law valid for small deviations from the reference trajectory. Such methods include the work of Kalman,[1] and Bryson and Denham.[2] Although Kalman considered the linear control problem, his results are applicable to the linearized motion about a reference trajectory of a nonlinear system. In the works just mentioned, the reference trajectory is arbitrary and need not be optimal in any sense.

A different approach was taken by Kelley,[3] Breakwell, Speyer, and Bryson,[4] Dreyfus[5] and this author.[6] These investigators obtained the linear feedback gains for what may be termed neighboring extremal control. Here the reference trajectory of the basic nonlinear system is optimal, and the linear control law is optimal in the same sense to a first approximation.

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Recently, Silber[7] developed a control scheme, in which the Lagrange multipliers are considered as control parameters, and obtained differential equations for the coefficients in the Taylor series expansion of the Lagrange multipliers as functions of the state. This permits a high-order approximation to neighboring extremals. Kushner[8] treats the stochastic optimal feedback control problem and presents a method of determining high-order corrections to the optimal deterministic control in the presence of stochastic disturbances.

In the present work we assume that the state of the nonlinear system can be measured exactly. Deviations in the state from an optimal nominal state are taken as the feedback information. We develop a high-order approximation to the nonlinear optimal feedback control law in the vicinity of an optimal nominal trajectory, which is a solution of the Mayer problem in variational calculus. The control scheme is a neighboring extremal control scheme in that it provides terminal control and, in the presence of disturbances, enables the system to follow trajectories that are approximately optimal in the same sense as the nominal trajectory to any order. Differential equations for the linear and quadratic optimal feedback gains are explicitly given. Computational aspects associated with these equations will be treated in a subsequent paper. The linear feedback gains are identical to those developed in Refs. 3–6.

Closed-form expressions for the linear and quadratic optimal feedback control gains for a simple intercept problem are